Charge & Current densities :

Klein - Gordon eq^n for a free particle is given by :-

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} - \frac{m^2 c^2}{\hbar^2}\psi = 0 \qquad (10)$$

taking complex conjugate, we get

$$\nabla^2 \psi^* - \frac{1}{c^2}\frac{\partial^2 \psi^*}{\partial t^2} - \frac{m^2 c^2}{\hbar^2}\psi^* = 0 \qquad (11)$$

multiply eq^n (10) & (11) by ψ^* & ψ respectively, we get

$$\psi^* \nabla^2 \psi - \frac{1}{c^2}\psi^* \frac{\partial^2 \psi}{\partial t^2} - \frac{m^2 c^2}{\hbar^2}\psi^* \psi = 0 \qquad (12)$$

$$\psi \nabla^2 \psi^* - \frac{1}{c^2} \psi \frac{\partial^2 \psi^*}{\partial t^2} - \frac{m^2 c^2}{\hbar^2} \psi^* \psi = 0 \quad (13)$$

Subtracting (13) from (12), we get

$$\Rightarrow \psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* - \frac{1}{c^2}\left[\psi^* \frac{\partial^2 \psi}{\partial t^2} - \psi \frac{\partial^2 \psi^*}{\partial t^2}\right] = 0$$

$$\Rightarrow \nabla \cdot \left[\psi^* \nabla \psi - \psi \nabla \psi^*\right] - \frac{1}{c^2} \frac{\partial}{\partial t}\left[\psi^* \frac{\partial \psi}{\partial t} - \frac{\partial \psi^*}{\partial t} \psi\right] = 0$$

multiply throughout by $\frac{\hbar}{2im}$, we get

$$\nabla \cdot \left[\frac{\hbar}{2im}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)\right] + \frac{\partial}{\partial t}\left[\frac{\hbar}{2imc^2}\left(\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t}\right)\right] = 0 \quad (14)$$

Substituting,

Probability density or charge density

$$P(r,t) = \frac{\hbar}{2imc^2}\left[\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t}\right] \quad (a)$$

current density

$$\text{and } S(r,t) = \frac{\hbar}{2im}\left[\psi^* \nabla \psi - \psi \nabla \psi^*\right] \quad (b)$$

(15)

eq^n (14) becomes :-

$$\boxed{\nabla \cdot S(r,t) + \frac{\partial \rho}{\partial t}(r,t) = 0} \quad (15)$$

This is the eq^n of continuity.

This shows clearly that the probability of finding the particle in any system remains conserved.

Here,

1. The current density S has the same form as in Non-relativistic case.

2. But the probability density or charge density ρ is not same as in Non-relativistic case, $\rho = \psi^* \psi$, due to following reasons:-

$$\rho(r,t) = \frac{\hbar}{2imc^2}\left[\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t}\right]$$

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$$\rho(r,t) = \frac{1}{2mc^2}\left[\left(-i\hbar \frac{\partial \psi^*}{\partial t}\right)\psi + \psi^*\left(-i\hbar \frac{\partial \psi}{\partial t}\right)\right] \quad — (17)$$

$$\rho(r,t) = \frac{1}{2mc^2}\left[(E\psi^*)\psi + \psi^*(E\psi)\right]$$

$$= \frac{1}{2mc^2}\left[2E\psi^*\psi\right]$$

$$\boxed{\rho(r,t) = \frac{E}{mc^2}\left[\psi^*\psi\right]} \quad — (18)$$

where $E = \pm\sqrt{(p^2c^2 + m^2c^4)}$

Difficulties associated with the interpretation of K-G eq^n :-

Since, $E = \pm\sqrt{p^2c^2 + m^2c^4}$ i.e., the total energy of the particle is +ve as well as negative.

This means that the probability $\rho(r, t)$ is not always +ve. Hence, it is no

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Same as conventional positional probability density ($\psi^* \psi$) i.e; ψ is not the amplitude probability as obtained by Schroedinger eq^n.

This means that the wave func^n ψ is not same as used in Schroedinger eq^n. This ψ is known as Field not the wave func^n.

According to new ~~impor~~ interpretation of ψ by Pauli-Wesskopf (1934), P multiplied by e i.e (eP) is interpretated as charge

eP = charge density or electronic charge density,

e may be +ve or -ve.

therefore eP can be positive and negative

eS = current density or electronic current density.

In K-G eq^n ψ has one component (scalar) i.e; ψ is real. If ψ has more components (more degree of freedom) than these components will show spin

spin → degrees of freedom (up, down, no spin) 11.
K.G → 1 d.f. → scalar (spinless)
(only orbital exists)

motion. In the absence of other components K-G eqn will describe particles of zero spin. like π-mesons.

Positive and negative sign shows that the relativistic theories give rise to both particles and anti-particles.

If the ~~en~~ energy '$E < mc^2$', then the probability P in K-G eqn reduces to non-relativistic probability density $\psi^* \psi$ as follows :-

density.

The wave soln is given by:-

$$\Psi(r,t) = \psi(r)\, e^{-iEt/\hbar}$$

If E' is non-relativistic energy, then total energy E may be expressed as $E = E' + mc^2$; mc^2 being rest energy

$$\psi(r,t) = \Psi(r)\, e^{-i(E'+mc^2)t/\hbar}$$

$$= \psi(r)\, e^{-E't/\hbar} \cdot e^{-imc^2 t/\hbar}$$

$$\psi(r,t) = \psi'(r,t)\, e^{-imc^2 t/\hbar} \quad (19)$$

differentiate (19) w.r.t 't'

$$\frac{\partial \psi}{\partial t} = \left(\frac{\partial \psi'}{\partial t} - \frac{imc^2}{\hbar} \psi' \right) e^{-imc^2 t/\hbar} \quad (20)$$

Taking complex conjugate of (19) and (20), we get

$$\psi^*(r,t) = \psi'^*(r,t)\, e^{icm^2 t/\hbar}$$

$$\frac{\partial \psi^*}{\partial t} = \left(\frac{\partial \psi'^*}{\partial t} + \frac{imc^2}{\hbar} \psi'^* \right) e^{imc^2 t/\hbar} \quad (21)$$

The probability is given by:-

$$P(r,t) = \frac{\hbar}{2imc^2} \left(\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t} \right)$$

Substituting these values, we get

$$P(r,t) = \frac{\hbar}{2imc^2} \Big\{ \psi'(r,t)\, e^{-imc^2 t/\hbar} \Big\} \Big\{ \frac{\partial \psi'^*}{}$$

$$\left.\frac{imc^2}{\hbar}\psi'^*\right) e^{imc^2t/\hbar}\Big\} - \Big\{\psi'^*(r,t)\, e^{imc^2t/\hbar}\Big\}$$

$$\Big\{\left(\frac{\partial\psi'}{\partial t} - \frac{imc^2}{\hbar}\psi'\right) e^{-imc^2t/\hbar}\Big\}\Bigg]$$

$$= \frac{\hbar}{2imc^2}\left[\psi'(r,t)\left(\frac{\partial\psi'}{\partial t} + \frac{imc^2}{\hbar}\psi'^*\right) - \psi'^*(r,t)\left(\frac{\partial\psi'}{\partial t} - \frac{imc^2}{\hbar}\psi'\right)\right]$$

$$= \frac{1}{2mc^2}\left[\psi'(r,t)\left(\frac{\hbar}{i}\frac{\partial\psi'^*}{\partial t}\right) - \psi'^*\left(i\hbar\frac{\partial\psi'}{\partial t}\right)\right] + \psi'^*\psi'$$

$$= \frac{1}{2mc^2}\left[\psi'(r,t)\left(-i\hbar\frac{\partial\psi'^*}{\partial t}\right) + \psi'^*\left(i\hbar\frac{\partial\psi'}{\partial t}\right)\right] + \psi'^*\psi'$$

$$= \frac{E'}{mc^2}\psi'^*\psi' + \psi'^*\psi'$$

(Since $E'^* = E'$, eigen value of H being real)

$$= \psi'^*\psi'$$

(Since non relativistic energy $E' << mc^2$)

which is correct non-relativistic expression for probability density.